UDC 532.135

CALCULATION OF INTERNAL FRICTION FOR A TWO-LAYER CYLINDER IN TORSIONAL VIBRATION

T. D. Shermergor and S. I. Meshkov

Inzhenerno-Fizicheskii Zhurnal, Vol. 9, No. 2, pp. 261–266, 1965

The internal friction associated with torsional oscillations of a cylindrical rod whose dynamic elastic modulus varies along the radius is examined. An effective rheological model and the internal friction of a two-layer rod are discussed.

We shall examine the torsional oscillations of a multilayer cylindrical rod, the n-th layer of which has shear modulus μ_n^*.

We shall consider the shear modulus to be a linear integral operator defined by the equality

$$\mu_n^* \varphi = \mu_n \int_{-\infty}^{t} f_n(t-t')\, \dot{\varphi}(t')\, dt'^*. \tag{1}$$

We shall examine only oscillations of small amplitude. Then, taking into account the hypothesis of plane sections and neglecting distortion of the radius, we find that the only nonzero component of the stress tensor $\sigma_{\theta z}$ in layer n is given by

$$\sigma^n_{\theta z} = 2\mu_n^* \varepsilon_{\theta z} = \mu_n^* r \partial\varphi/\partial z. \tag{2}$$

Using (2), we find the component $dM_{\theta r}$ of the moment of the forces in layer n of thickness dz:

$$dM^n_{\theta r} = \frac{\pi}{2}\,(r_n^4 - r_{n-1}^4)\,\mu_n^* \frac{\partial^2\varphi}{\partial z^2}\, dz. \tag{3}$$

Denoting the radius of the specimen by R and taking into account that the moment of the inertia forces of layer dz is $(\pi/2) R^4 \rho \ddot{\varphi}\, dz$, we obtain the following equation of motion:

$$\rho\ddot{\varphi} = \sum_n \mu_n^* R^{-4} (r_n^4 - r_{n-1}^4)\, \partial^2\varphi/\partial z^2. \tag{4}$$

Hence the effective dynamic modulus of the multilayer cylinder will be

$$\mu^* = R^{-4} \sum_n \mu_n^* (r_n^4 - r_{n-1}^4). \tag{5}$$

In the case of continuous variation of the properties along the radius, Eq. (5) may be written in integral form:

$$\mu^* = 4R^{-4} \int_0^R \mu^*(r)\, r^3 dr. \tag{6}$$

Formual (5) or (6) may be used to calculate the internal friction in a multilayer rod with arbitrary memory function. We shall restrict our attention to the simplest case, when the rod has two layers and the memory function is described by the sum of exponentials:

$$f_n(t) = \sum_\alpha \nu_{n\alpha} \exp(-t/\tau_{n\alpha}), \tag{7}$$

where $\nu_{n\alpha}$ gives the contribution of the given relaxation time to the total effect, and $0 \leqslant \nu_{n\alpha} \leqslant 1$ Then, for forced harmonic oscillations $\varphi = \varphi_0 \exp i\omega t$, operator μ_n^* degenerates to the function

$$\mu_n(\omega) = \mu_n \sum_\alpha \nu_{n\alpha} \frac{i\omega\tau_{n\alpha}}{1 + i\omega\tau_{n\alpha}}. \tag{8}$$

* A dot over the φ indicates differentiation with respect to time.

The effective shear modulus $\mu(\omega)$, in our case of a two-layer cylinder, will be expressed as

$$\mu(\omega) = \frac{r_0^4}{R^4}\mu_1 \sum_{\alpha=1}^{N_1} \nu_{1\alpha} \frac{i\omega\tau_{1\alpha}}{1+i\omega\tau_{1\alpha}} + \left(1 - \frac{r_0^4}{R^4}\right)\mu_2 \sum_{\alpha=1}^{N_2} \nu_{2\alpha} \frac{i\omega\tau_{2\alpha}}{1+i\omega\tau_{2\alpha}}. \tag{9}$$

Hence the internal friction is determined as the tangent of the phase shift angle between shear stress and strain according to the usual formula

$$\operatorname{tg}\delta = \operatorname{Gm}\mu(\omega)/\operatorname{Re}\mu(\omega).$$

From the general expression (9), it is easy to determine the internal friction for various rheological properties of the inner and outer zones. For example, if the inner zone is elastic and the outer layer Maxwellian, then in (9) we should put

$$N_1 = N_2 = 1,\ \nu_{\alpha 1} = \nu_{\alpha 2} = 1,\ \tau_1 = \infty,\ \tau_2 = \tau.$$

Then

$$\mu(\omega) = \mu_0 + (\mu_\infty - \mu_0)\frac{i\omega\tau}{1+i\omega\tau}, \tag{10}$$

where

$$\mu_0 = \mu_1 r_0^4/R^4;\ \mu_\infty - \mu_0 = \mu_2(1 - r_0^4/R^4). \tag{11}$$

Formula (10) coincides with the well-known expression for a standard linear body [1], and for the internal friction $\operatorname{tg}\delta$ gives the expression

$$\operatorname{tg}\delta = \Delta_\mu \omega\tau_0/(1+\omega^2\tau_0^2), \tag{12}$$

Results of Calculations for Simple Rheological Models*

Model	Complex modulus	Equivalent model	Complex modulus	Notation
A B	μ_1 $\mu_2\frac{i\omega\tau}{1+i\omega\tau}$	C	$\mu_0 + \Delta\mu\frac{i\omega\tau}{1+i\omega\tau}$	$\mu_0 = \mu_1 r_0^4/R^4;$ $\Delta\mu = \mu_2(1 - r_0^4/R^4)$
B A	$\mu_1\frac{i\omega\tau}{1+i\omega\tau}$ μ_2	C	$\mu_0 + \Delta\mu\frac{i\omega\tau}{1+i\omega\tau}$	$\mu_0 = \mu_2(1 - r_0^4/R^4)$ $\Delta\mu = \mu_1 r_0^4/R^4$
A C	μ_1 $\mu_2^0 + \Delta\mu_2\frac{i\omega\tau}{1+i\omega\tau}$	C	$\mu_0 + \Delta\mu\frac{i\omega\tau}{1+i\omega\tau}$	$\mu_0 = \mu_1 r_0^4/R^4 +$ $+\mu_2^0(1 - r_0^4/R^4);$ $\Delta\mu = \Delta\mu_2\left(1 - \frac{r_0^4}{R^4}\right)$
C A	$\mu_1^0 + \Delta\mu_1\frac{i\omega\tau}{1+i\omega\tau}$ μ_2	C	$\mu_0 + \Delta\mu\frac{i\omega\tau}{1+i\omega\tau}$	$\mu_0 = \mu_2(1 - r_0^4/R^4) +$ $+\mu_1^0 r_0^4/R^4;$ $\Delta\mu = \Delta\mu_1 r_0^4/R^4$
B B	$\mu_1\frac{i\omega\tau_1}{1+i\omega\tau_1}$ $\mu_2\frac{i\omega\tau_2}{1+i\omega\tau_2}$	D	$\bar{\mu}_1\frac{i\omega\tau_1}{1+i\omega\tau_1} +$ $+\bar{\mu}_2\frac{i\omega\tau_2}{1+i\omega\tau_2}$	$\bar{\mu}_1 = \mu_1 r_0^4/R^4;$ $\bar{\mu}_2 = \mu_2(1 - r_0^4/R^4)$

* The first line refers to the core, the second to the outer layer. A is the elastic model, B the Maxwellian, C the standard linear body, and D the four-element model (a successive combination of the Maxwell and Voigt models).

where the modulus defect and the new relaxation time are expressed as

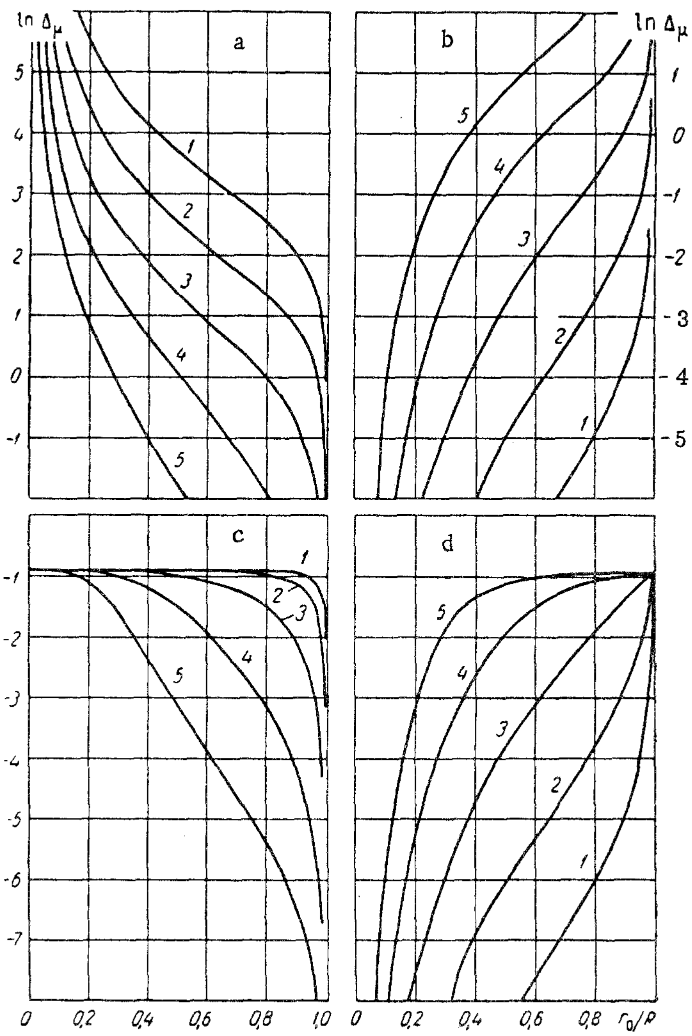
$$\Delta_\mu = (\mu_\infty - \mu_0)/\sqrt{\mu_\infty \mu_0}; \quad \tau_0 = \tau \sqrt{\mu_\infty/\mu_0}. \tag{13}$$

The calculation is similar for other memory properties of both zones. The result of such a calculation for simple rheological models is presented in the table.

It can be seen from (12) that the intensity of internal friction of the standard linear body is characterized by the modulus defect Δ_μ, which determines the maximum internal friction $\mathrm{tg}\,\delta_{max} = \Delta_\mu/2$. Thus, by way of a graphical illustration of the influence of the ratio of elastic moduli and of outer layer thickness on the internal friction, Fig. 1a shows the dependence of modulus defect on the ratio of radii r_0/R for various μ_1/μ_2.

It can be seen from the curves given that the first series (Maxwellian outer layer) is characterized, in general, by a higher level of internal friction than for a Maxwellian core. This is understandable if we consider that the torsion strain is nonuniform and the surface layers are strongly deformed. Therefore, in the first case, energy dissipation occurs mainly in the zone of high energy concentration, and, in the second, in the zone of low energy concentration.

Similar curves are shown in Fig. 1b for modulus defect in the case when the inner zone is elastic, and the outer is described by the equation of the standard linear body, and vice versa. In the calculation it is assumed that $\mu_0 = (2/3) \times \mu_\infty$, i.e., that one third of the total shear modulus relaxes. As might also be expected, the presence of an elastic zone decreases the internal friction, this decrease being the greater, the larger the shear modulus for the elastic zone.



Dependence of logarithm of modulus defect on core thickness r_0/R of a two-layer cylinder: a) elastic core and Maxwellian outer layer; b) vice versa; c) internal zone elastic and outer zone described by the equation of the standard linear body; d) vice versa; 1) $\mu_1/\mu_2 = 10^{-2}$, 2) 10^{-1}, 3) 1, 4) 10, 5) 10^2.

If the radial nonuniformity of the specimen is due to the action of surface-active agents, these substances may inhibit the relaxation processes in the surface layer; for example, total suppression of the relaxation peak has been observed in diffusion saturation of tin with copper [2]. Then, as follows from Fig. 1d, for the corresponding boundary layer thickness the internal friction may be appreciably changed. Surface-active agents, however, may have a substantial influence on the internal friction not only by hindering or facilitating the course of various of the relaxation processes, but also by changing the elastic modulus of the surface layers or of the specimen as a whole.

For a two-layer specimen, the internal friction may also be calculated by another method. We represent the energy losses ΔW in the specimen as the sum of losses in the individual layers $\Delta W = \Delta W_1 + \Delta W_2$ Then, using the expression for internal friction $Q^{-1} = \Delta W/2\pi W$, we obtain

$$Q^{-1} = \frac{W_1}{W} Q_1^{-1} + \frac{W_2}{W} Q_2^{-1} = Q_1^{-1} + \frac{W_2}{W}(Q_2^{-1} - Q_1^{-1}). \tag{14}$$

The ratio of the energies W_2/W is found from the known formulas of elasticity theory, which give

$$\frac{W}{W_2} = 1 + \frac{\mu_1 r_0^4}{\mu_2 (R^4 - r_0^4)}. \tag{15}$$

An experimental verification of (14), taking (15) into account, was carried out in [3] for constant frequency and temperature; the author confirmed the strong dependence of internal friction of a two-layer specimen on the ratio of the elastic moduli.

If for the internal friction Q^{-1} in (14), we take the corresponding expression for Q^{-1} based on the rheological properties of the layer, and also take into account that the moduli μ_i in (15) must be replaced by the dynamic moduli $\mathrm{Re}\,\mu_i(\omega)$, we can verify that the same result is obtained for internal friction as when (9) is used.

Formulas (14) and (15) are convenient for describing the internal friction in a two-layer torsional pendulum. However, if the properties of the specimen vary continuously along the radius, and also when dealing with other memory phenomena, creep and stress relaxation, in the approximation considered, the direct formulas (5) or (6) should be used. Moreover, in describing high internal friction associated with damped oscillations, when the corresponding boundary problem [4, 5] must be solved, Eq. (4) should be used, along with an effective modulus determined from (1), (5), and (7).

NOTATION

μ_n – shear modulus of the n-th layer (nonrelaxation modulus); μ_n^* – shear modulus operator of the n-th layer; μ^* – effective shear modulus operator for the cylinder; $\mu_n(\omega)$ – complex elastic modulus of the n-th layer; $\mu(\omega)$ – complex effective elastic modulus of the cylinder; f_n – memory function; φ – torsion angle; σ_{ik}, ε_{ik} – stress and strain tensors; $\varepsilon_{\vartheta z}$ – component of strain tensor; r_n and R – radii of layer and specimen; r_0 – radius of internal cylindrical zone; $\tau_{n\alpha}$ – relaxation times; $\nu_{n\alpha}$ – coefficients determining the intensity of a given relaxation process; μ_0, μ_∞ – relaxation and nonrelaxation elastic moduli; $\Delta\mu$ – modulus defect; W – energy; Q^{-1} – internal friction.

REFERENCES

1. C. M. Zener, collection: Elasticity and Anelasticity of Metals, ed. S. V. Vonsovskii [Russian translation], IL, 1954.

2. E. P. Zakoshchikova and V. S. Postnikov, collection: Relaxation Phenomena in Metals and Alloys, ed. V. S. Postnikov [in Russian], Metallurgizdat, 1963.

3. Yu. K. Favstov, Izv. AN SSSR, OTN, Mekhanika i mashinostroenie, no. 3, 1963.

4. S. I. Meshkov and T. D Shermergor, Zhurnal prikladnoi mekhaniki i tekhnicheskoi fiziki, no. 6, 1962.

5. T. D. Shermergor and S. I. Meshkov, IFZh, no. 5, 1963.

24 November 1964

Voronezh Polytechnic Institute